

# On finite generation of self-similar groups of finite type

Ievgen V. Bondarenko, Igor O. Samoilovych

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## Abstract

A self-similar group of finite type is the profinite group of all automorphisms of a regular rooted tree that locally around every vertex act as elements of a given finite group of allowed actions. We provide criteria for determining when a self-similar group of finite type is finite, level-transitive, or topologically finitely generated. Using these criteria and GAP computations we show that for the binary alphabet there is no infinite topologically finitely generated self-similar group given by patterns of depth 3, and there are 32 such groups for depth 4.

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## 1 Introduction

There are two important classes of groups acting on regular rooted trees that have arisen as a generalization of the Grigorchuk group: self-similar groups and branch groups. An automorphism group of a regular rooted tree is self-similar if the restriction of the action of every its element onto every subtree can be given again by an element of the group. There are many examples of self-similar groups with numerous extreme properties (like the Grigorchuk group) and this class of groups is very promising for looking different counterexamples. At the same time, self-similar groups appear naturally in many areas of mathematics and have strong connections with fractal geometry, dynamical systems, automata theory (see [8] and the references therein). Branch groups are automorphism groups of a tree whose subgroup lattice is similar to the tree [1]. This class plays an important role in classification of just-infinite groups [5].

Self-similar groups of finite type have arisen as the closure of certain self-similar branch groups in the topology of the tree. It was noticed in [4, Section 7] that the closure of the Grigorchuk group is a profinite self-similar group that can be described by a finite group of allowed local actions on a finite tree (obtained from the binary tree by truncating at some depth). R.I. Grigorchuk used this observation to define a self-similar group of finite type

as the group of all tree automorphisms that locally around every vertex act as elements of a given finite group (see precise definition in the next section). The term “group of finite type” comes from the analogy with shifts of finite type in symbolic dynamics [7] (note that a different term, namely finitely constrained group, is used in [9, 10]). Every self-similar group of finite type with transitive action on levels of the tree is a profinite branch group by [4, Proposition 7.5], and conversely, the closure (and profinite completion) of a self-similar regular branch group with congruence subgroup property is a self-similar group of finite type by [9, Theorem 3]. The last observation was the main ingredient to compute the Hausdorff dimension of such branch groups in [9].

Although a self-similar group of finite type is easy to define by a finite group of patterns, it is not clear what are the properties of the group. In particular, R.I. Grigorchuk asked in [4, Problem 7.3(i)] under what conditions a self-similar group of finite type is topologically finitely generated. In this note we address this question and establish certain criterion in Theorem 3 as well as some necessary and sufficient conditions. We also answer such basic questions like how to check whether a self-similar group of finite type is trivial, finite, or acts transitively on levels of the tree.

The closure of the Grigorchuk group is a self-similar group of finite type defined by patterns of depth 4 over the binary tree. The closure of groups defined in [9] give examples of infinite finitely generated self-similar groups of finite type defined by patterns of depth  $d$  for any  $d \geq 4$ . For depth 2 and binary tree every self-similar group of finite type is either finite or not finitely generated as shown in [10]. The only unknown case was for depth 3. Using developed criteria and GAP computations we prove that there is no infinite finitely generated self-similar group of finite type defined by patterns of depth 3 over the binary tree. For depth 4 there are 32 such groups (including the closures of the Grigorchuk group and the iterated monodromy group of  $z^2 + i$  [6]).

## 2 Self-similar groups of finite type

In this section we first recall all needed definitions and introduce self-similar groups of finite type (see [8, 4] for more information). After that we study conditions when a self-similar group of finite type is trivial, finite, or level-transitive.

**Tree  $X^*$ .** Let  $X$  be a finite alphabet with at least two letters. Let  $X^*$  be the free monoid freely generated by  $X$ . The elements of  $X^*$  are all finite words  $x_1x_2 \dots x_n$  over  $X$  (including the empty word). We also use notation  $X^*$  for the tree with the vertex set  $X^*$  and edges  $(v, vx)$  for all  $v \in X^*$  and  $x \in X$ . The set  $X^n$  is the  $n$ -th level of the tree  $X^*$ . The subtree of  $X^*$  induced by the set of vertices  $\cup_{i=0}^n X^i$  is denoted by  $X^{[n]}$ .

Self-similar groups of finite type are defined as special subgroups of the group  $\text{Aut } X^*$  of all automorphisms of the tree  $X^*$ . The group  $\text{Aut } X^*$  is profinite; it is the inverse limit of the sequence

$$\dots \rightarrow \text{Aut } X^{[3]} \rightarrow \text{Aut } X^{[2]} \rightarrow \text{Aut } X,$$

where the homomorphisms are given by restriction of the action.

**Sections of automorphisms.** For every automorphism  $g \in \text{Aut } X^*$  and every word  $v \in X^*$  define the *section*  $g_{(v)} \in \text{Aut } X^*$  of  $g$  at  $v$  by the rule:  $g_{(v)}(x) = y$  for  $x, y \in X^*$  if and only if  $g(vx) = g(v)y$ . In other words, the section of  $g$  at  $v$  is the unique automorphism  $g_{(v)}$  of  $X^*$  such that  $g(vx) = g(v)g_{(v)}(x)$  for all  $x \in X^*$ . Sections have the following properties:

$$g_{(vu)} = g_{(v)}(u), \quad (g^{-1})_{(v)} = (g_{(g^{-1}(v))})^{-1}, \quad (gh)_{(v)} = g_{(h(v))}h_{(v)}$$

for all  $v, u \in X^*$  (we are using left actions, i.e.,  $(gh)(v) = g(h(v))$ ).

A subgroup  $G < \text{Aut } X^*$  is called *self-similar* if  $g_{(v)} \in G$  for every  $g \in G$  and  $v \in X^*$ .

The restriction of the action of an automorphism  $g \in \text{Aut } X^*$  to the subtree  $X^{[d]}$  is denoted by  $g|_{X^{[d]}} \in \text{Aut } X^{[d]}$ . To every  $g \in \text{Aut } X^*$  there corresponds a collection  $(g_{(v)}|_{X^{[d]}})_{v \in X^*}$  of automorphisms from  $\text{Aut } X^{[d]}$  which completely describe the action of  $g$  on the tree  $X^*$ .

**Self-similar groups of finite type.** A subgroup  $\mathcal{P}$  of  $\text{Aut } X^{[d]}$  will be called a *group of patterns of depth  $d$*  (or a pattern group of depth  $d$ ),  $d \geq 1$ . We say that an automorphism  $g \in \text{Aut } X^*$  *agrees with  $\mathcal{P}$*  if every section  $g_{(v)}$ ,  $v \in X^*$ , acts on  $X^{[d]}$  in the same way as some element in  $\mathcal{P}$ , i.e.,  $g_{(v)}|_{X^{[d]}} \in \mathcal{P}$  for all  $v \in X^*$ . Since  $\mathcal{P}$  is a group, the inverse  $g^{-1}$  and all sections  $g_{(v)}$  of such an element  $g$  agree with  $\mathcal{P}$ , the product of two elements that agree with  $\mathcal{P}$  also agrees with  $\mathcal{P}$ . We obtain the self-similar group  $G_{\mathcal{P}}$  of all automorphisms  $g \in \text{Aut } X^*$  that agree with  $\mathcal{P}$ , i.e., we define the group

$$G_{\mathcal{P}} = \{g \in \text{Aut } X^* : g_{(v)}|_{X^{[d]}} \in \mathcal{P} \text{ for every } v \in X^*\},$$

called the *self-similar group of finite type given by the pattern group  $\mathcal{P}$* . Note that Grigorchuk in [4] introduced these groups using finite sets of forbidden patterns, while we are using “allowed” patterns.

Every group  $G_{\mathcal{P}}$  is closed in the topology of  $\text{Aut } X^*$ . Indeed, if for an element  $g \in \text{Aut } X^*$  the restriction  $g|_{X^{[n]}}$  belongs to  $G_{\mathcal{P}}|_{X^{[n]}}$  for every  $n \in \mathbb{N}$ , then  $g_{(v)}|_{X^{[d]}} \in \mathcal{P}$  for every  $v \in X^*$  and thus  $g \in G_{\mathcal{P}}$ . Hence  $G_{\mathcal{P}}$  is a profinite group.

Let us consider a few simple examples. If  $\mathcal{P}$  is trivial then  $G_{\mathcal{P}}$  is trivial. If  $\mathcal{P} = \text{Aut } X^{[d]}$  then  $G_{\mathcal{P}} = \text{Aut } X^*$  (for any  $d \in \mathbb{N}$ ). For every subgroup  $\mathcal{P} < \text{Sym}(X)$  the infinitely iterated permutational wreath product  $\dots \wr_X \mathcal{P} \wr_X \mathcal{P}$  is a self-similar group of finite type, where  $\mathcal{P}$  is the corresponding group of patterns of depth 1, and every self-similar group of finite type given by patterns of depth 1 is of this form.

**Minimal pattern groups.** The same self-similar group of finite type may be given by different groups of patterns of depth  $d$  and we want to choose a unique pattern group in each class. Let  $G$  be a self-similar group of finite type given by a group of patterns of depth  $d$  and consider the pattern group  $\mathcal{P} = G|_{X^{[d]}}$ . Since the group  $G$  is self-similar,  $g_{(v)}|_{X^{[d]}} \in \mathcal{P}$  for every  $g \in G$  and  $v \in X^*$ , and thus  $G < G_{\mathcal{P}}$ . On the other hand, it is clear from the definition that every pattern group of depth  $d$  that produces  $G$  contains  $\mathcal{P}$  as a subgroup. Hence  $G = G_{\mathcal{P}}$  and  $\mathcal{P}$  is the smallest group of patterns of depth  $d$  with this property. A pattern group  $\mathcal{P}$  of depth  $d$  will be called *minimal* if the equality  $G_{\mathcal{P}} = G_{\mathcal{Q}}$  for  $\mathcal{Q} < \text{Aut } X^{[d]}$  implies  $\mathcal{P} < \mathcal{Q}$ . It follows from the above arguments that a pattern group  $\mathcal{P}$

of depth  $d$  is minimal if and only if  $\mathcal{P} = G_{\mathcal{P}}|_{X^{[d]}}$ , in other words, if every pattern in  $\mathcal{P}$  is realized as a restriction of an element of  $G_{\mathcal{P}}$ . Every self-similar group of finite type given by patterns of depth  $d$  is represented by a unique minimal pattern group of depth  $d$ .

**Pattern graph.** Let  $\mathcal{P}$  be a group of patterns of depth  $d$ . In order to minimize  $\mathcal{P}$  one can use a directed labeled graph  $\Gamma_{\mathcal{P}}$  which we call the *pattern graph* associated to  $\mathcal{P}$ . The vertices of  $\Gamma_{\mathcal{P}}$  are the elements of  $\mathcal{P}$  and for  $a, b \in \mathcal{P}$  and  $x \in X$  we put a labeled arrow  $a \xrightarrow{x} b$  whenever  $a_{(x)}|_{X^{[d-1]}} = b|_{X^{[d-1]}}$ . Informally, the arrow  $a \xrightarrow{x} b$  shows that we can use the pattern  $b$  to extend the action of  $a$  on the subtree  $xX^{[d]}$  (see Fig. 1). If a vertex  $a \in \mathcal{P}$  does not have an outgoing edge labeled by  $x$  for some letter  $x \in X$ , then the action of  $a$  cannot be extended to the next level using patterns from  $\mathcal{P}$ ; in other words  $a$  is not a restriction of an element of  $G_{\mathcal{P}}$ . Now it is clear how to minimize  $\mathcal{P}$ : we just remove every vertex that does not have an outgoing edge labeled by  $x$  for some  $x \in X$  and repeat this reduction as long as possible. The remaining patterns will form a minimal pattern group for  $G_{\mathcal{P}}$ . In particular,  $\mathcal{P}$  is minimal if and only if every vertex of  $\Gamma_{\mathcal{P}}$  has an outgoing edge labeled by  $x$  for every  $x \in X$ .

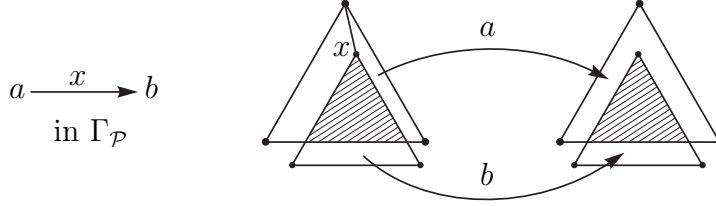


Figure 1: Coordination between patterns.

The graph  $\Gamma_{\mathcal{P}}$  can be used to represent elements of the group  $G_{\mathcal{P}}$  by graph homomorphisms as follows. Let us take the tree  $X^*$  and add direction and label to every edge by  $v \xrightarrow{x} vx$  for every  $v \in X^*$  and  $x \in X$ . Then every element  $g \in G_{\mathcal{P}}$  defines a homomorphism  $\phi : X^* \rightarrow \Gamma_{\mathcal{P}}$  of labeled directed graphs by the rule  $\phi(v) = g_{(v)}|_{X^{[d]}}$ . Indeed, for every arrow  $v \xrightarrow{x} vx$  in the tree  $X^*$  the elements  $g_{(v)(x)}$  and  $g_{(vx)}$  are the same and we have the arrow  $\phi(v) \xrightarrow{x} \phi(vx)$  in the graph  $\Gamma_{\mathcal{P}}$ . And vice versa, every homomorphism  $\phi : X^* \rightarrow \Gamma_{\mathcal{P}}$  defines an element  $g \in G_{\mathcal{P}}$  by its restrictions  $g_{(v)}|_{X^{[d]}} = \phi(v)$ . This description is an analog of a standard statement in symbolic dynamics that every shift of finite type is sofic (see [7, Theorem 3.1.5]), and pattern graphs play a role of recognition graphs. One can use this observation to introduce the notion of a self-similar group of sofic type which we will discuss elsewhere.

**Branching properties.** Let us explain the connection between self-similar groups of finite type and branch groups mentioned in Introduction.

Let  $G$  be a subgroup of  $\text{Aut } X^*$ . The *vertex stabilizer*  $\text{St}_G(v)$  of a vertex  $v \in X^*$  is the subgroup of all  $g \in G$  such that  $g(v) = v$ . The  *$n$ -th level stabilizer*  $\text{St}_G(n)$  is the subgroup of all  $g \in G$  such that  $g(v) = v$  for every  $v \in X^n$ . Notice that  $\text{St}_G(v)$  and  $\text{St}_G(n)$  have finite index in  $G$ . The *rigid vertex stabilizer*  $\text{RiSt}_G(v)$  of a vertex  $v \in X^*$  is the subgroup

of all  $g \in G$  such that  $g(u) = u$  for every vertex  $u \in X^* \setminus vX^*$ . The set of all sections  $g_{(v)}$  for  $g \in \text{RiSt}_G(v)$  forms a group which we call the *section group* of  $\text{RiSt}_G(v)$  at the vertex  $v$ . The group  $G$  is called *level-transitive* if it acts transitively on all levels  $X^n$  of the tree. The group  $G$  is called *regular branch* branching over its subgroup  $K$  if  $G$  is level-transitive,  $K$  is a normal subgroup of finite index, and the group of all automorphism  $g \in \text{St}_{\text{Aut } X^*}(1)$  such that the tuple  $(g_{(x)})_{x \in X}$  belongs to  $\prod_X K$  is a subgroup of finite index in  $K$ . Note that the last condition implies that the section group of  $\text{RiSt}_K(v)$  at  $v$  contains  $K$  for every vertex  $v \in X^*$ .

Every level-transitive self-similar group  $G_{\mathcal{P}}$  of finite type given by patterns of depth  $d$  is regular branch over its level stabilizer  $\text{St}_{G_{\mathcal{P}}}(d-1)$  (see [4, Proposition 7.15]). Indeed, notice that for every element  $h \in \text{St}_{G_{\mathcal{P}}}(d-1)$  and any vertex  $v \in X^*$  the unique automorphism  $g \in \text{RiSt}_{\text{Aut } X^*}(v)$  such that  $g_{(v)} = h$  agrees with the pattern group  $\mathcal{P}$  and hence belongs to  $G_{\mathcal{P}}$ . It follows that  $\text{St}_{G_{\mathcal{P}}}(n)$  for  $n \geq d$  decomposes into the direct product

$$\text{St}_{G_{\mathcal{P}}}(n) \cong \text{St}_{G_{\mathcal{P}}}(d-1) \times \dots \times \text{St}_{G_{\mathcal{P}}}(d-1)$$

of  $|X|^{n-d+1}$  copies of  $\text{St}_{G_{\mathcal{P}}}(d-1)$ , where each factor acts on the corresponding subtree  $vX^*$  for  $v \in X^{n-d+1}$ . The last condition in the definition of a regular branch group follows. Conversely, if  $G$  is a self-similar regular branch group branching over its level stabilizer  $\text{St}_G(d-1)$  then the closure of  $G$  in  $\text{Aut } X^*$  is a self-similar group of finite type given by patterns of depth  $d$  (see [9, Theorem 3]).

**Triviality, finiteness, and level-transitivity of  $G_{\mathcal{P}}$ .** Given a pattern group  $\mathcal{P}$  we want to understand whether the group  $G_{\mathcal{P}}$  is trivial, finite, or acts transitively on the levels of the tree  $X^*$ . The answer to the question about triviality of  $G_{\mathcal{P}}$  directly follows from the definition of a minimal pattern group. Namely, the group  $G_{\mathcal{P}}$  is trivial if and only if minimizing  $\mathcal{P}$  we obtain the trivial group.

The finiteness of  $G_{\mathcal{P}}$  can be effectively checked using the next statement.

**Proposition 1.** *Let  $\mathcal{P}$  be a minimal pattern group of depth  $d$ . The group  $G_{\mathcal{P}}$  is finite if and only if the stabilizer  $\text{St}_{\mathcal{P}}(d-1)$  is trivial, and in this case  $G_{\mathcal{P}}$  is isomorphic to  $\mathcal{P}$ .*

*Proof.* Let  $\Gamma_{\mathcal{P}}$  be the pattern graph of  $\mathcal{P}$  and put  $m = |\text{St}_{\mathcal{P}}(d-1)|$ . Notice that  $(bc)|_{X^{[d-1]}} = b|_{X^{[d-1]}}$  for every  $b \in \mathcal{P}$  and  $c \in \text{St}_{\mathcal{P}}(d-1)$ . Hence if  $a \xrightarrow{x} b$  is an arrow in  $\Gamma_{\mathcal{P}}$  then  $a \xrightarrow{x} bc$  is also an arrow in  $\Gamma_{\mathcal{P}}$  for every  $c \in \text{St}_{\mathcal{P}}(d-1)$ , and every outgoing arrow at  $a$  with label  $x$  is of this form. Therefore, since  $\mathcal{P}$  is minimal, every vertex of  $\Gamma_{\mathcal{P}}$  has precisely  $m$  outgoing edges labeled by  $x$  for every  $x \in X$ . It follows that for every  $a \in \mathcal{P}$  there are precisely  $m^{|X|}$  elements  $g \in G_{\mathcal{P}}|_{X^{[d+1]}}$  such that  $g|_{X^{[d]}} = a$ ; in other words, every pattern in  $\mathcal{P}$  has  $m^{|X|}$  extensions to the next level. Then for each level  $n > d$  and for every  $f \in G_{\mathcal{P}}|_{X^{[n]}}$  there are precisely  $m^{|X|^{n-d+1}}$  elements  $g \in G_{\mathcal{P}}|_{X^{[n+1]}}$  such that  $g|_{X^{[n]}} = f$ . Now we can compute the total number of elements in the restriction  $G_{\mathcal{P}}|_{X^{[n]}}$ :

$$|G_{\mathcal{P}}|_{X^{[n]}}| = |\mathcal{P}| \cdot m^{|X| + |X|^2 + \dots + |X|^{n-d}}, \text{ for } n > d.$$

Therefore the group  $G_{\mathcal{P}}$  is finite if and only if  $m = 1$ , i.e., when the group  $\text{St}_{\mathcal{P}}(d-1)$  is trivial. In this case,  $|G_{\mathcal{P}}| = |\mathcal{P}|$  and the restriction  $g \mapsto g|_{X^{[d]}}$  is an isomorphism between  $G_{\mathcal{P}}$  and  $\mathcal{P}$ .  $\square$

It follows from the proof that we can also use the pattern graph  $\Gamma_{\mathcal{P}}$  to check the finiteness of  $G_{\mathcal{P}}$ . If  $\mathcal{P}$  is minimal, then the group  $G_{\mathcal{P}}$  is finite if and only if some (equivalently, every) vertex of  $\Gamma_{\mathcal{P}}$  has only one outgoing edge labeled by  $x$  for each  $x \in X$ .

Let us treat transitivity on levels. We will use the standard observation that a subgroup  $G < \text{Aut } X^*$  acts transitively on  $X^{n+1}$  if and only if it acts transitively on  $X^n$  and the stabilizer  $\text{St}_G(v)$  of some (every) vertex  $v \in X^n$  acts transitively on  $vX$ .

Let  $\mathcal{P}$  be a minimal pattern group of depth  $d$  and consider the self-similar group of finite type  $G_{\mathcal{P}}$ . We fix a letter  $x \in X$  and use notation  $x^n$  for the word  $x \dots x$  ( $n$  times). Let  $\mathcal{P}_n$  be the group of all elements  $a \in \mathcal{P}$  for which there exists  $g \in \text{St}_{G_{\mathcal{P}}}(x^n)$  such that  $g(x^n)|_{X^{[d]}} = a$ . Then  $\text{St}_{G_{\mathcal{P}}}(x^n)$  is transitive on  $x^n X$  if and only if  $\mathcal{P}_n$  is transitive on  $X$ . It follows that  $G_{\mathcal{P}}$  is level-transitive if and only if each group  $\mathcal{P}_n$  for  $n \geq 0$  is transitive on  $X$ . Notice that the groups  $\mathcal{P}_n$  can be computed recursively by the rule:  $\mathcal{P}_0 = \mathcal{P}$  and

$$\mathcal{P}_{n+1} = \{a \in \mathcal{P}_n : \text{there exists } b \in \text{St}_{\mathcal{P}_n}(x) \text{ such that } b_{(x)}|_{X^{[d-1]}} = a|_{X^{[d-1]}}\}.$$

We obtain a decreasing sequence  $\mathcal{P} > \mathcal{P}_1 > \dots$  of finite groups which should stabilize on some subgroup  $\mathcal{Q} < \mathcal{P}$ ,  $\mathcal{Q} = \bigcap_{n \geq 0} \mathcal{P}_n$ . Moreover, if we take the smallest  $n$  such that  $\mathcal{P}_n = \mathcal{P}_{n+1}$  then  $\mathcal{P}_n = \mathcal{P}_{n+k}$  for every  $k \in \mathbb{N}$ , and thus  $\mathcal{Q} = \mathcal{P}_n$ . Hence the group  $\mathcal{Q}$  can be algorithmically computed. We have proved the following effective criterium.

**Proposition 2.** *The group  $G_{\mathcal{P}}$  is level-transitive if and only if the group  $\mathcal{Q}$  is transitive on  $X$ .*

### 3 Finite generation of groups $G_{\mathcal{P}}$

In this section we study when the group  $G_{\mathcal{P}}$  is topologically finitely generated. Further we omit the word “topologically”.

**Theorem 3.** *Let  $G$  be a level-transitive self-similar group of finite type given by patterns of depth  $d$ . The group  $G$  is finitely generated if and only if there exists  $n \geq d$  such that the commutator of  $\text{St}_G(d-1)|_{X^{[n]}}$  contains  $\text{St}_G(n-1)|_{X^{[n]}}$ .*

*Proof.* Let  $G = G_{\mathcal{P}}$  for a minimal pattern group  $\mathcal{P}$  of depth  $d$ .

First we prove the necessity. The proof will not use transitivity on levels. Assume that the commutator of  $\text{St}_G(d-1)|_{X^{[n]}}$  does not contain  $\text{St}_G(n-1)|_{X^{[n]}}$  for every  $n \geq d$ . Let us prove that  $\text{St}_G(d-1)$  and thus  $G$  are not finitely generated. In the proof we will use notations  $S = \text{St}_G(d-1)$  and  $S_n = S|_{X^{[n]}}$ . For each  $m \geq d$  consider the homomorphism

$$\varphi : S \rightarrow \prod_{n=d}^m S_n/[S_n, S_n], \quad \varphi(g) = (g|_{X^{[n]}}[S_n, S_n])_{n=d}^m.$$

Recall that the stabilizer  $\text{St}_G(n-1)|_{X^{[n]}}$  decomposes into the direct product  $\text{St}_{\mathcal{P}}(d-1) \times \dots \times \text{St}_{\mathcal{P}}(d-1)$  of  $|X|^{n-d}$  copies of  $\text{St}_{\mathcal{P}}(d-1)$ . By our assumption there exists an element  $g_n = (1, \dots, a_n, \dots, 1)$ ,  $a_n \in \text{St}_{\mathcal{P}}(d-1)$ , of this product that does not belong to the

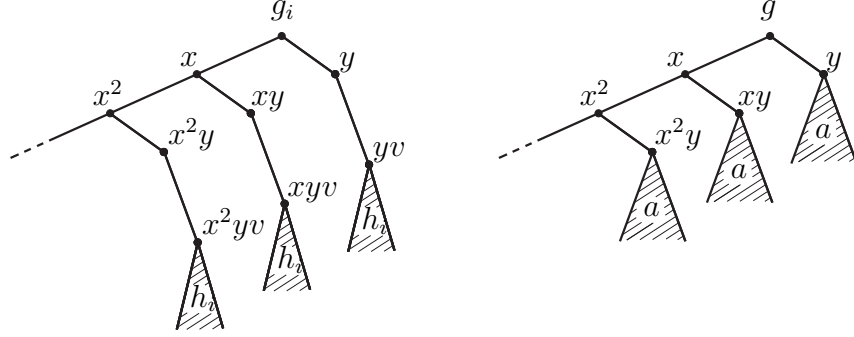


Figure 2: The construction of generator  $g_i$  and conjugator  $g$ .

commutator  $[S_n, S_n]$ . Let  $A_n$  be the group generated by the image of  $g_n$  in the quotient of  $\text{St}_G(n-1)|_{X^{[n]}}$  by  $[S_n, S_n]$ . The group  $A_n$  is a nontrivial subgroup of the finite abelian group  $S_n/[S_n, S_n]$ . Hence  $A_n$  is also a quotient of  $S_n/[S_n, S_n]$ . Composing with  $\varphi$  we obtain a homomorphism from  $S$  to  $\prod_{n=d}^m A_n$ . Moreover, for  $i < n$  the  $i$ -th component of the image of  $g_n$  in this direct product is trivial. It follows that  $\prod_{n=d}^m A_n$  is a homomorphic image of  $S$ . Since  $|A_n| \leq |\mathcal{P}|$  for all  $n$ , the number of generators of  $\prod_{n=d}^m A_n$  goes to infinity as  $m$  goes to infinity. Hence  $S$  is not finitely generated.

Let us prove the converse. Fix  $k \geq d$  such that the commutator of  $\text{St}_G(d-1)|_{X^{[k]}}$  contains  $\text{St}_G(k-1)|_{X^{[k]}}$ . We construct a finitely generated dense subgroup of  $G$  using the techniques from branch groups (see [1, 2]). Let  $f_1, \dots, f_l$  and  $h_1, \dots, h_m$  be the elements of  $G$  such that

$$\langle f_1, \dots, f_l \rangle|_{X^{[1+d+k]}} = G|_{X^{[1+d+k]}} \quad \text{and} \quad \langle h_1, \dots, h_m \rangle|_{X^{[k]}} = \text{St}_G(d-1)|_{X^{[k]}}.$$

The group  $\text{St}_G(d-1)$  is nontrivial by Proposition 1, and we can find  $v \in X^d$  and  $a \in \text{St}_G(d-1)$  such that  $a(v) \neq v$  (the element  $a$  will be used to shift the section of certain automorphisms at the vertex  $v$ ). Fix two letter  $x, y \in X$ ,  $x \neq y$ . Define the automorphisms  $g_1, \dots, g_m$  recursively by their sections:

$$g_i(yv) = h_i \quad \text{and} \quad g_i(x) = g_i, \quad i = 1, \dots, m,$$

and the other sections are trivial (see Fig. 2). Notice that  $g_1, \dots, g_m$  belong to  $G$ .

Consider the group  $H = \langle f_1, \dots, f_l, h_1, \dots, h_m, g_1, \dots, g_m \rangle$  and let us show that  $H$  is dense in  $G$ . We need to prove that  $H|_{X^{[n]}} = G|_{X^{[n]}}$  for all  $n \in \mathbb{N}$ . The statement holds for  $n \leq 1 + d + k$  by construction. By induction on  $n$  assume that we have proved it for all levels  $\leq n + d + k$ . There exists an element  $g \in G$  such that

$$g(x^i y) = a \quad \text{for } i = 0, \dots, n-1$$

and the other sections are trivial (see Fig. 2). By inductive hypothesis there exists  $h \in H$  such that  $h|_{X^{[n+d+k]}} = g|_{X^{[n+d+k]}}$ . Then the commutator  $[h^{-1}g_i h, g_j]$  acts trivially on the vertices in  $X^{[n+d+k+1]} \setminus x^n y v X^{[k]}$  and at the vertex  $x^n y v$  has section

$$[h^{-1}g_i h, g_j]_{(x^n y v)} = [h_{(x^n y v)}^{-1} h_i h_{(x^n y v)}, h_j].$$

Conjugating by generators  $g_1, \dots, g_m$  we obtain that the section group of  $\text{RiSt}_H(x^n y v)|_{X^{[n+d+k+1]}}$  at  $x^n y v$  contains the commutator of  $\text{St}_G(d-1)|_{X^{[k]}}$  and hence  $\text{St}_G(k-1)|_{X^{[k]}}$ . Since the action is transitive this holds for every vertex of the level  $X^{n+d+1}$ . Hence  $\text{St}_H(n+d+k)|_{X^{[n+d+k+1]}} = \text{St}_G(n+d+k)|_{X^{[n+d+k+1]}}$  and the statement follows.  $\square$

**Remark 1.** An automorphism of the tree  $X^*$  is called *finite-state* if it has finitely many sections (the term comes from automata theory); a subgroup is finite-state if it consists of finite-state automorphisms. We can always choose elements  $f_1, \dots, f_l$  and  $h_1, \dots, h_m$  so that they are finite-state. Then the elements  $g_1, \dots, g_m$  and the group  $H$  constructed in the proof will be also finite-state. Adding sections of elements we obtain a finitely generated finite-state self-similar dense subgroup in  $G$ .

**Remark 2.** The condition of level-transitivity cannot be dropped in Theorem 3. For example, consider the alternating group  $A_5$  with the natural action on  $\{1, 2, 3, 4, 5\}$ , extend the action to the alphabet  $X = \{0, 1, 2, 3, 4, 5\}$  by putting  $\pi(0) = 0$  for every  $\pi \in A_5$ , and consider the infinitely iterated permutational wreath product  $G_{A_5} = \dots \wr_X A_5 \wr_X A_5$ . The group  $A_5$  is perfect, i.e.,  $[A_5, A_5] = A_5$ , hence the condition in Theorem 3 holds for  $n = d = 1$ . However the group  $G$  is not finitely generated, because the map  $g \mapsto (g|_{X^{(0^n)}})|_X)_{n \in \mathbb{N}}$  is a surjective homomorphism from  $G$  to the product  $\prod_{\mathbb{N}} A_5$  which is not finitely generated.

**Remark 3.** It is not difficult to see that for a group  $G_{\mathcal{P}}$  given by a transitive pattern group  $\mathcal{P}$  of depth 1 the condition in the theorem holds for some  $n$  if and only if the group  $\mathcal{P}$  is perfect. Hence Theorem 3 generalizes Corollary 3.6 in [2] about finite generation of iterated permutational wreath products  $\dots \wr_X \mathcal{P} \wr_X \mathcal{P}$ .

**Proposition 4.** *Let  $G$  be a self-similar group of finite type given by patterns of depth  $d$ . If there exists  $n \geq d$  such that the commutator of  $G|_{X^{[n]}}$  does not contain  $\text{St}_G(n-1)|_{X^{[n]}}$  then the group  $G$  is not finitely generated.*

*Proof.* The proof uses the same arguments as in the first part of the proof above. Fix  $n \geq d$  such that the commutator of  $G_n := G|_{X^{[n]}}$  does not contain  $\text{St}_G(n-1)|_{X^{[n]}}$ . For every  $k \in \mathbb{N}$  consider the map

$$\varphi_k : G \rightarrow G_n/[G_n, G_n], \quad \varphi_k(g) = \prod_{v \in X^k} g(v)|_{X^{[n]}}[G_n, G_n].$$

Since  $G_n/[G_n, G_n]$  is abelian every map  $\varphi_k$  is a homomorphism. Now for every  $m \in \mathbb{N}$  consider the homomorphism  $\varphi : G \rightarrow \prod_{k=1}^m G_n/[G_n, G_n]$ ,  $\varphi(g) = (\varphi_k(g))_{k=1}^m$ . For every  $k$  and every pattern  $a \in \text{St}_G(n-1)|_{X^{[n]}}$  there exists  $g$  in the rigid stabilizer  $\text{RiSt}_G(v)$  of a vertex  $v \in X^k$  such that  $g(v)|_{X^{[n]}} = a$ , and thus  $\varphi_k(g) = a$  and  $\varphi_i(g) = e$  for  $i < k$ . Since  $\text{St}_G(n-1)|_{X^{[n]}}/[G_n, G_n]$  is a homomorphic image of  $G_n$ , it follows that the abelian group  $\prod_{k=1}^m \text{St}_G(n-1)|_{X^{[n]}}/[G_n, G_n]$  is a homomorphic image of  $G$  for every  $m$ . Hence  $G$  is not finitely generated.  $\square$

The next statement generalizes Proposition 2 in [10].



**Corollary 5.** *Let  $\mathcal{P}$  be an abelian pattern group. The group  $G_{\mathcal{P}}$  is finitely generated if and only if it is finite.*

*Proof.* The statement follows from Proposition 1 and Proposition 4 with  $n = d$ .  $\square$

**Corollary 6.** *Take a cyclic subgroup  $C < \text{Sym}(X)$  and consider the group  $C \wr_X C$  as a natural subgroup of  $\text{Aut } X^{[2]} \cong \text{Sym}(X) \wr_X \text{Sym}(X)$ . Then for any nilpotent pattern group  $\mathcal{P} < C \wr_X C$  the group  $G_{\mathcal{P}}$  is finitely generated if and only if it is finite.*

*Proof.* Since  $\mathcal{P}/\text{St}_{\mathcal{P}}(1)$  is cyclic, the commutator  $[\mathcal{P}, \mathcal{P}]$  is a subgroup of  $\text{St}_{\mathcal{P}}(1)$ . If it is a proper subgroup then the group  $G_{\mathcal{P}}$  is not finitely generated by Proposition 4. Suppose  $[\mathcal{P}, \mathcal{P}] = \text{St}_{\mathcal{P}}(1)$ . For any  $a, b \in \mathcal{P}$  there exists  $k \in \mathbb{N}$  such that  $a^k b$  or  $b^k a$  belongs to  $\text{St}_{\mathcal{P}}(1)$ . Using the equality  $[a, b] = [a, a^k b] = [b^k a, b]$  we obtain that  $[\mathcal{P}, \mathcal{P}] = [\mathcal{P}, \text{St}_{\mathcal{P}}(1)]$ . Since  $\mathcal{P}$  is nilpotent, the last equality implies that  $[\mathcal{P}, \mathcal{P}] = \text{St}_{\mathcal{P}}(1) = \{1\}$  and hence the group  $G_{\mathcal{P}}$  is finite by Proposition 1.  $\square$

## 4 A few classification results

In this section we classify self-similar groups of finite type for the binary alphabet  $X = \{0, 1\}$  and depth  $\leq 4$ . All computations were made in GAP. Our strategy for classifying self-similar groups of finite type of a given depth  $d$  is the following. First we find all subgroups in  $\text{Aut } X^{[d]}$ , then minimize all subgroups and obtain the number of all minimal pattern groups, which is equal to the number of self-similar groups of finite type of a given depth as subgroups in  $\text{Aut } X^*$ . Further we distinguish all finite groups using Proposition 1. Then we apply Proposition 4 for small values of  $n$  to distinguish groups that are not finitely generated. An infinite self-similar group over the binary alphabet is level-transitive (see [3, Lemma 3]), hence the rest of the groups are level-transitive and we can apply Theorem 3. In this way it was possible to obtain the following results.

**Depth  $d = 2$ .** This case was treated in [10]. There are ten subgroups in  $\text{Aut } X^{[2]}$ , six minimal pattern subgroups, and hence six self-similar groups of finite type. Among them there are three finite groups, namely the trivial group and two groups isomorphic to  $C_2$ , and the other three groups are not finitely generated (Proposition 4 works with  $n = 2$ ).

**Depth  $d = 3$ .** There are 576 subgroups in  $\text{Aut } X^{[3]}$ , 60 minimal pattern subgroups, and hence 60 self-similar groups of finite type. Among them there are 23 finite groups, namely the trivial group, two groups isomorphic to  $C_2$ , four groups isomorphic to  $C_2 \times C_2$ , 16 groups isomorphic to the dihedral group  $D_8$ . The other 37 groups are not finitely generated (27 groups satisfy Proposition 4 with  $n = 3$  and 10 groups with  $n = 4$ ).

**Corollary 7.** *A self-similar group of finite type given by patterns of depth  $d \leq 3$  over the binary alphabet is either finite or not finitely generated.*

**Depth  $d = 4$ .** There are 4544 self-similar groups of finite type. Among them there are 1535 finite groups, namely the trivial group, two groups isomorphic to  $C_2$ , four groups isomorphic to  $C_2 \times C_2$ , 16 groups isomorphic to  $D_8$ , eight groups isomorphic to  $C_2 \times C_2 \times C_2$ ,

96 groups isomorphic to  $C_2 \times D_8$ , 128 groups isomorphic to  $(C_2 \times C_2 \times C_2 \times C_2) \rtimes C_2$ , 256 groups isomorphic to  $((C_4 \times C_2) \rtimes C_2) \rtimes C_2$ , and 1024 groups isomorphic to  $\text{Aut } X^{[3]} \cong C_2 \wr_X C_2 \wr_X C_2$ . Among the rest of the groups there are 2977 not finitely generated (1235 groups satisfy Proposition 4 with  $n = 4$ , 778 groups with  $n = 5$ , 508 groups with  $n = 6$ , 200 groups with  $n = 7$ , and 256 groups with  $n = 8$ ) and 32 finitely generated groups that satisfy Theorem 3 with  $n = 6$ . The pattern groups of these 32 self-similar groups of finite type all have order 4096, their restriction on  $X^{[3]}$  is equal to  $\text{Aut } X^{[3]}$ , and among them there are 20 pairwise non-isomorphic groups. These pattern groups can be described as follows. Let us consider the group  $\text{Aut } X^{[4]}$  as a natural subgroup of the symmetric group  $\text{Sym}(16)$  on the set  $\{1, 2, \dots, 16\} \leftrightarrow X^4$  and fix the permutations:

$$\begin{aligned} a_1 &= (1, 9)(2, 10)(3, 11)(4, 12)(5, 13)(6, 14)(7, 15)(8, 16) \\ a_2 &= (1, 10, 2, 9)(3, 11)(4, 12)(5, 14, 6, 13)(7, 15)(8, 16) \\ a_3 &= (1, 10)(2, 9)(3, 11)(4, 12)(5, 13)(6, 14)(7, 15)(8, 16) \\ a_4 &= (1, 9, 2, 10)(3, 11)(4, 12)(5, 14, 6, 13)(7, 15)(8, 16) \end{aligned}$$

$$\begin{aligned} b_1 &= (1, 5)(2, 6)(3, 7)(4, 8)(9, 10) & c_1 &= (1, 3)(2, 4) & c_3 &= (1, 3)(2, 4)(5, 6) \\ b_2 &= (1, 6)(2, 5)(3, 7)(4, 8)(9, 10) & c_2 &= (1, 4, 2, 3) & c_4 &= (1, 4, 2, 3)(5, 6) \end{aligned}$$

Then the 32 pattern groups mentioned above is the family of groups  $\mathcal{P}_{ijk} = \langle a_i, b_j, c_k \rangle$ . In this family: the self-similar group of finite type  $G_{\mathcal{P}_{123}}$  is the closure of the Grigorchuk group and  $G_{\mathcal{P}_{111}}$  is the closure of the iterated monodromy group of  $z^2 + i$  [6].

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